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Léonard GALLARDO* and **Chaabane REJEB†**

Abstract

In this paper we show that the Dunkl intertwining operator has a compact support which is invariant by the associated Coxeter-Weyl group. This property enables us to determine explicitely the support of the volume mean value operator, a fundamental tool for the study of harmonic functions relative to the Dunkl-Laplacian operator.

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1 Introduction and statement of the results

Let R be a (finite) root system in $\mathbb{R}^d$ with associated Coxeter-Weyl group W (see [7] or [9] for details on root systems) and for $\xi \in \mathbb{R}^d$, let D_ξ be the Dunkl operator defined by

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^d),$$

where R_+ is a subsystem of positive roots, σ_α is the reflection directed by the root $\alpha \in R_+$, k is a nonnegative multiplicity function (defined on R) and $\partial_\xi f$ is the usual ξ-directional derivative of f.

These operators, introduced by C. F. Dunkl in the nineties (see [1]), are related to partial derivatives by means of an intertwining operator V_k (see [3] or [4]) as follows

$$\forall\ \xi \in \mathbb{R}^d, \quad D_\xi V_k = V_k \partial_\xi. \tag{1.1}$$

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We know that V_k is a topological isomorphism from the space $\mathcal{C}^\infty(\mathbb{R}^d)$ (carrying its usual Fréchet topology) onto itself satisfying (1.1) and $V_k(1) = 1$ (see [15]) and V_k commutes with the W-action (see [14]) i.e.

$$\forall \;\; f \in \mathcal{C}^\infty(\mathbb{R}^d), \;\; \forall \;\; g \in W, \;\; g^{-1}.V_k(g.f) = V_k(f), \tag{1.2}$$

where $g.f(x) = f(g^{-1}x)$.

A fundamental fact due to M. Rösler (see [11] or [14]) is that for every $x \in \mathbb{R}^d$, there exists a unique compactly supported probability measure μ_x^k on $\mathbb{R}^d$ with

$$\text{supp } \mu_x^k \subset C(x) := co\{gx, \; g \in W\} \tag{1.3}$$

(the convex hull of the orbit of x under the group W) such that

$$\forall \; f \in \mathcal{C}^\infty(\mathbb{R}^d), \;\; V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x^k(y). \tag{1.4}$$

Note that the property (1.3) has been proved in [8].
Throughout this paper, the notation $k > 0$ means that $k(\alpha) > 0$ for all $\alpha \in R$.

Concerning the measure μ_x^k (which we call Rösler's measure at point x), the first result of our paper is the following

Theorem A. For every $x \in \mathbb{R}^d$, we have
1) $x \in \text{supp } \mu_x^k$,

2) If $k > 0$, the support of μ_x^k is a W-invariant set,

3) If $k > 0$ then $W.x$ (the W-orbit of x) is contained in supp μ_x^k.

A question strongly related to the support of Rösler's measures concerns the Dunkl-mean value operator introduced by the authors in [6] in the study of harmonic functions for the Dunkl-Laplacian operator $\Delta_k = \sum_{i=1}^d D_i^2$ where $D_i = D_{e_i}$ with $(e_i)_{1 \leq i \leq d}$ an orthonormal basis of $\mathbb{R}^d$. Precisely for $x \in \mathbb{R}^d$ and $r > 0$, the mean value of a continuous function f at (x, r) is defined by

$$M_B^r(f)(x) := \frac{1}{m_k(B(0,r))} \int_{\mathbb{R}^d} f(y) h_k(r, x, y) \omega_k(y) dy,$$

where $y \mapsto h_k(r, x, y)$ is a compactly supported measurable function (see Section 2) given by

$$h_k(r, x, y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle}) d\mu_y^k(z), \tag{1.5}$$

m_k is the measure $dm_k(x) := \omega_k(x)dx$ and ω_k is the weight function

$$\omega_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}. \tag{1.6}$$

In particular we have shown that a $\mathcal{C}^2(\mathbb{R}^d)$-function u is Δ_k-harmonic in $\mathbb{R}^d$ if and only if for all $(x,r) \in \mathbb{R}^d \times \mathbb{R}_+$, $u(x) = M_B^r(u)(x)$. For a further thorough study of Δ_k-harmonicity on a general W-invariant open set, it would be crucial to get information on the support of the mean value operators. We already know that the measures

$$d\eta_{x,r}^k = \frac{1}{m_k(B(0,r))} h_k(r,x,y)\omega_k(y)dy \quad (x \in \mathbb{R}^d, r > 0), \tag{1.7}$$

are probability measures with compact support equal to supp $h_k(r,x,.)$ and satisfying the following inclusion ([6]):

$$\text{supp } h_k(r,x,.\,) \subset B^W(x,r) := \cup_{g\in W} B(gx,r), \tag{1.8}$$

where $B(x,r)$ denotes the usual closed ball of radius r centered at x.

In fact, the second main result of this paper, intimately related to Theorem A, is a precise description of the support of $h_k(r,x,.)$. It states that

Theorem B: Let $x \in \mathbb{R}^d$ and $r > 0$.

1) We have $B(x,r) \subset \text{supp } h_k(r,x,.)$.

2) If $k > 0$, then we have

$$\text{supp } h_k(r,x,.) = B^W(x,r) := \cup_{g\in W} B(g.x,r).$$

We will call $B^W(x,r)$ the closed Dunkl ball centered at x and with radius $r > 0$ associated to the Coxeter-Weyl group W.

2 The harmonic kernel and the mean value operator

In this section we recall some results of [6].

Let $(r,x,y) \mapsto h_k(r,x,y)$ be the harmonic kernel defined by (1.5). We note that in the classical case (i.e. $k = 0$), we have $\mu_y^k = \delta_y$ and $h_0(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$. The harmonic kernel satisfies the following properties (see [6]):

1. For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \le h_k(r,x,y) \le 1$.

2. For all fixed $x, y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r,x,y)$ is right-continuous and non decreasing on $]0,+\infty[$.

3. Let $r > 0$ and $x \in \mathbb{R}^d$. For any sequence $(\varphi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions such that for every $\varepsilon > 0$,

$$0 \le \varphi_\varepsilon \le 1,\ \varphi_\varepsilon = 1 \text{ on } B(0,r) \ \text{ and } \forall\, y \in \mathbb{R}^d, \ \ \lim_{\varepsilon\to 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y),$$

we have

$$\forall\, y \in \mathbb{R}^d, \quad h_k(r,x,y) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \widetilde{\varphi}_\varepsilon(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle})d\mu_y^k(z),$$

where $\widetilde{\varphi}_\varepsilon$ is the profile function of φ_ε i.e. $\varphi_\varepsilon(x) = \widetilde{\varphi}_\varepsilon(\|x\|)$.

4. For all $r > 0$, $x, y \in \mathbb{R}^d$ and $g \in W$, we have

$$h_k(r, x, y) = h_k(r, y, x) \quad \text{and} \quad h_k(r, gx, y) = h_k(r, x, g^{-1}y). \tag{2.1}$$

5. For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r, x, .)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r, x, y)\omega_k(y)dy = m_k(B(0, r)) = \frac{d_k r^{d+2\gamma}}{d + 2\gamma}, \tag{2.2}$$

where d_k is the constant

$$d_k := \int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2+\gamma)}.$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of $\mathbb{R}^d$ and c_k is the Macdonald-Mehta constant (see [10], [5]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x)dx.$$

6. Let $r > 0$ and $x \in \mathbb{R}^d$. Then the function $h_k(r, x, .)$ is upper semi-continuous on $\mathbb{R}^d$.

7. The harmonic kernel satisfies the following geometric inequality: if $\|a - b\| \leq 2r$ with $r > 0$, then

$$\forall\, \xi \in \mathbb{R}^d, \quad h_k(r, a, \xi) \leq h_k(4r, b, \xi)$$

(see [6], Lemma 4.1). Note that in the classical case (i.e. $k = 0$), this inequality says that if $\|a - b\| \leq 2r$, then $B(a, r) \subset B(b, 4r)$.

8. Let $x \in \mathbb{R}^d$. Then the family of probability measures $d\eta^k_{x,r}(y)$ defined by (1.7) is an approximation of the Dirac measure δ_x as $r \longrightarrow 0$. That is

$$\forall\, \alpha > 0, \quad \lim_{r \to 0} \int_{\|x-y\|>\alpha} d\eta^k_{x,r}(y) = 0$$

and if f is a locally bounded measurable function on a W-invariant open neighborhood of x and if f is continuous at x, then (see [6], Proposition 3.2):

$$\lim_{r \to 0} \int_{\mathbb{R}^d} f(y)d\eta^k_{x,r}(y) = \lim_{r \to 0} M^r_B(f)(x) = f(x). \tag{2.3}$$

3 Proof of the results

For convenience we group together the first points of Theorem A and Theorem B in the following Proposition.

Proposition 3.1 *Let $x \in \mathbb{R}^d$. Then*

i) *for every $r > 0$, $x \in$ supp $h_k(r, x, .)$,*

ii) $x \in$ *supp* μ_x^k,

iii) *for every* $r > 0$, $B(x,r) \subset$ *supp* $h_k(r,x,.)$.

Proof: **i)** Suppose that there exists $r > 0$ such that $x \notin \text{supp } h_k(r,x,.)$. Then we can find $\varepsilon > 0$ such that $h_k(r,x,y) = 0$, for all $y \in B(x,\varepsilon)$. Let f be a nonnegative continuous functions on $\mathbb{R}^d$ such that supp $f \subset B(x,\varepsilon)$ and $f = 1$ on $B(x,\varepsilon/2)$.
Since $t \mapsto h_k(t,x,y)$ is increasing on $]0,+\infty[$, we deduce that

$$\forall\ t \in]0,r], \quad 0 \leq M_B^t(f)(x) \leq \frac{1}{m_k[B(0,t)]} \int_{\mathbb{R}^d} f(y) h_k(r,x,y)\omega_k(y)dy = 0.$$

Hence, we obtain $M_B^t(f)(x) = 0$, for all $t \in]0,r]$. Letting $t \to 0$ and using the relation (2.3), we get a contradiction.

ii) Let $x \in \mathbb{R}^d$ be fixed. At first, we claim that

$$\forall\ r > 0, \quad \forall\ y \in \mathbb{R}^d, \quad h_k(r,x,y) \leq \mu_x^k[B(y,r)]. \tag{3.1}$$

Indeed, from the inclusion supp $\mu_x^k \subset B(0,\|x\|)$, we see that

$$\forall\ y \in \mathbb{R}^d, \quad \forall\ z \in \text{supp } \mu_x^k, \quad \|y-z\|^2 \leq \|y\|^2 + \|x\|^2 - 2\langle y,z\rangle.$$

This implies for any $y \in \mathbb{R}^d$ and $r > 0$ that

$$\forall\ z \in \text{supp } \mu_x^k, \quad \mathbf{1}_{[0,r]}\big(\sqrt{\|y\|^2 + \|x\|^2 - 2\langle y,z\rangle}\big) \leq \mathbf{1}_{[0,r]}(\|y-z\|) = \mathbf{1}_{B(y,r)}(z).$$

If we integrate the two terms of the previous inequality with respect to the measure μ_x^k, we obtain $h_k(r,y,x) \leq \mu_x^k\big(B(y,r)\big)$ and then (3.1) follows from (2.1).
Now, if $x \notin$ supp μ_x^k, there exists $\epsilon > 0$ such that $\mu_x^k\big(B(x,\epsilon)\big) = 0$. Thus, we have $\mu_x^k\big(B(y,\epsilon/2)\big) = 0$ whenever $y \in B(x,\epsilon/2)$. Using (3.1), we deduce that $h_k(\epsilon/2,x,.) = 0$ on $B(x,\epsilon/2)$, a contradiction with the result of i).

iii) Let $y \in \mathbb{R}^d$ such that $\|x-y\| < r$. As $\lim_{z\to y}(\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle) = \|x-y\|^2$, there exists $\eta > 0$ such that

$$\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle} \leq r \quad \text{for every } z \in B(y,\eta).$$

Therefore, by using the fact that $y \in$ supp μ_y^k we obtain $h_k(r,x,y) \geq \mu_y^k[B(y,\eta)] > 0$. $\square$

Remark 3.1 *For* $\alpha \in R$, *let*

$$H_\alpha := \{x \in \mathbb{R}^d,\ \langle x,\alpha\rangle = 0\}$$

be the hyperplane directed by α. *Note that in [12] (Corollary 3.6) and under the condition* $x \notin \cup_{\alpha\in R} H_\alpha$, *Rösler has proved that* $x \in$ *supp* μ_x^k *by using the asymptotic behavior of the Dunkl kernel* $E_k(x,y)$ *which is defined by*

$$E_k(x,y) := V_k\big(e^{\langle .,y\rangle}\big)(x) = \int_{\mathbb{R}^d} e^{\langle z,y\rangle} d\mu_x^k(z).$$

We turn now to the second statement of Theorem A that we recall below:

Theorem 3.1 *Let $x \in \mathbb{R}^d$ and assume that $k > 0$. Then the set supp μ_x^k is W-invariant.*

Proof: In order to simplify the formulas, we will assume here that the root system R is normalized i.e. $\|\alpha\|^2 = 2$ for all $\alpha \in R$. In particular, for reflections we have $\sigma_\alpha x = x - \langle \alpha, x\rangle\, \alpha$.
We will prove that if $y \in$ supp μ_x^k, then $\sigma_\alpha y \in$ supp μ_x^k for every $\alpha \in R$. Let then $y \in$ supp μ_x^k and suppose that there is a root $\alpha \in R$ such that $\sigma_\alpha y \notin$ supp μ_x^k. Write $y' := \sigma_\alpha y$ to simplify notations. There is a ball $B(y', \epsilon)$ ($\epsilon > 0$) such that for all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ with compact support included in $B(y', \epsilon)$, we have

$$\textstyle\int_{\mathbb{R}^d} f(z)\mu_x(dz) = V_k f(x) = 0.$$

Let us denote by $C^\infty_{y',\epsilon}$ (resp. $C_{y',\epsilon}$) the set of all functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ (resp. $f \in \mathcal{C}(\mathbb{R}^d)$) with compact support in $B(y', \epsilon)$. For all $\xi \in \mathbb{R}^d$ and all $f \in C^\infty_{y',\epsilon}$, we also have $\partial_\xi f \in C^\infty_{y',\epsilon}$. By the intertwining relation (1.1) we obtain

$$\forall\ \xi \in \mathbb{R}^d, \quad \forall\ f \in C^\infty_{y',\epsilon}, \quad D_\xi V_k f(x) = 0.$$

Suppose $f \in C^\infty_{y',\epsilon}$ and $f \geq 0$ and let $g := V_k f$. We have $g \geq 0$ on $\mathbb{R}^d$ (because V_k preserves positivity) and

$$\forall\ \xi \in \mathbb{R}^d, \quad D_\xi g(x) = \partial_\xi g(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{g(x) - g(\sigma_\alpha x)}{\langle x, \alpha\rangle} = 0. \tag{3.2}$$

But as $g(x) = 0$, x is a minimum of g so $\partial_\xi g(x) = 0$ and relation (3.2) implies

$$\forall\ \xi \in \mathbb{R}^d, \quad \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{g(x) - g(\sigma_\alpha x)}{\langle x, \alpha\rangle} = 0. \tag{3.3}$$

Now, consider the set

$$R_x := \{\alpha \in R_+;\ x \in H_\alpha\}.$$

There are two possible locations for x:
• **First case**: Suppose that $R_x = \emptyset$ i.e $x \notin \cup_{\alpha \in R} H_\alpha$ (i.e. for all root $\alpha \in R$, $\langle x, \alpha\rangle \neq 0$). Applying (3.3) with $\xi = x$ and using the fact that $g(x) = 0$, we get

$$\textstyle\sum_{\alpha \in R_+} k(\alpha) g(\sigma_\alpha x) = 0.$$

As $g \geq 0$ and $k > 0$, we obtain that $g(\sigma_\alpha x) = V_k f(\sigma_\alpha x) = 0$ for all $\alpha \in R_+$ and all $f \in C^\infty_{y',\epsilon}$ and $f \geq 0$. By uniform approximation, we deduce that for all $f \in C_{y',\epsilon}$ and $f \geq 0$, we also have $V_k f(\sigma_\alpha x) = 0$. Finally for every $f \in C_{y',\epsilon}$, by decomposing $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ and using the linearity and W-equivariance of V_k (relation (1.2)), we obtain that

$$\forall\ f \in C_{y',\epsilon}, \quad \forall\ \alpha \in R_+, \quad V_k f(\sigma_\alpha x) = V_k(\sigma_\alpha.f)(x) = 0,$$

where $\sigma_\alpha.f$ is the function $z \mapsto f(\sigma_\alpha z)$. As it is easy to see that $\sigma_\alpha.C_{y',\epsilon} = C_{\sigma_\alpha y',\epsilon}$, we deduce that

$$\forall\ \alpha \in R_+, \quad \forall f \in C_{\sigma_\alpha y',\epsilon}, \quad V_k f(x) = 0.$$

But this implies in particular that $V_k f(x) = 0$ for all $f \in C_{y,\epsilon}$ in contradiction with the hypothesis $y \in \text{supp}\ \mu_x^k$. The result of the theorem follows in the first case.

• **Second case**: Suppose that $R_x \neq \emptyset$. For every $\beta \in R_x$, clearly we have $x = \sigma_\beta x$. Therefore, since $g(x) = 0$, we get $g(\sigma_\beta x) = 0$, for all $\beta \in R_x$. But, as x is a minimum of g, we have

$$\forall\ \beta \in R_x, \quad \frac{g(x) - g(\sigma_\beta x)}{\langle x, \beta\rangle} = \int_0^1 \partial_\beta g(x - t\langle x, \beta\rangle \beta)dt = \partial_\beta g(x) = 0.$$

Hence, the relation (3.3) with $\xi = x$ implies

$$\textstyle\sum_{\alpha \in R_+ \setminus R_x} k(\alpha) g(\sigma_\alpha x) = 0.$$

Consequently, we obtain $g(\sigma_\alpha x) = 0$ for all $\alpha \in R$. The end of the proof of the first case applies and gives also the result in this case. This completes the proof of the theorem. □

From the W-invariance property of the support of μ_x^k and the fact that $x \in \text{supp}\ \mu_x^k$, we obtain immediately the last assertion of Theorem A:

Corollary 3.1 *Let $x \in \mathbb{R}^d$ and assume that $k > 0$. Then, for all $g \in W$, $gx \in$ supp μ_x^k.*

Now, we can turn to the proof of the second statement of Theorem B.

Corollary 3.2 *Let $x \in \mathbb{R}^d$ and $r > 0$. If $k > 0$, then*

$$\text{supp}\ h_k(r, x, .) = B^W(x, r) := \cup_{g \in W} B(gx, r). \tag{3.4}$$

Proof: Let $g \in W$ and $y \in \mathbb{R}^d$ such that $\|gx - y\| < r$. We will proceed as in the proof of Proposition 3.1, iii). We have

$$\lim_{z \to g^{-1}y} \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle} = \|x - g^{-1}y\|.$$

Hence, there exists $\eta > 0$ such that for all $z \in B(g^{-1}y, \eta)$, $\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle} \leq r$ and thus $h_k(r, x, y) \geq \mu_y^k[B(g^{-1}y, \eta)]$.
But, from the fact that $g^{-1}y \in \text{supp}\ \mu_y^k$ we deduce that $y \in \text{supp}\ h_k(r, x, .)$.
This completes the proof. □

Remark 3.2 *When $k \geq 0$, we will say that a root $\alpha \in R$ is active if $k(\alpha) > 0$. Let us denote by $R_A = \{\alpha \in R;\ k(\alpha) > 0\}$ the set of active roots and F the vector subspace of $\mathbb{R}^d$ generated by $\{\alpha,\ \alpha \in R_A\}$. Then we can generalize the results of Theorems A and B in the following form*

a) *The set* R_A *is a root system. Indeed, using the fact that* k *is* W*-invariant, we can see that for every* $\alpha,\beta\in R_A$, $k\big(\sigma_\alpha\beta\big)=k(\beta)>0$. *Thus*

$$\forall\ \alpha\in R_A,\quad R_A\cap\mathbb{R}\alpha=\{\pm\alpha\}\quad and\quad \sigma_\alpha(R_A)=R_A.$$

b) *Let* W_A *be the Coxeter-Weyl group associated to the root system* R_A. *Then the restriction* k_A *of* k *to* R_A *is clearly invariant under the* W_A*-action. In other words, it is a multiplicity function.*

c) *For any* $\xi\in\mathbb{R}^d$, *we will use the notation* $\xi=\xi'+\xi''\in F+F^\perp=\mathbb{R}^d$ *(where* $F^\perp$ *is the orthogonal complement of* F *in* $\mathbb{R}^d$*).*

• *Let* $x\in\mathbb{R}^d$. *Rösler's measure* μ_x^k *is of the form (see [13])*

$$\mu_x^k=\mu_{x'}^{k_A}\otimes\delta_{x''},\tag{3.5}$$

where $\mu_{x'}^{k_A}$ *is Rösler's measure associated to* (R_A,k_A) *and* $\delta_{x''}$ *is the Dirac measure at* x''. *We have*

$$supp\ \mu_x^k=x''+supp\ \mu_{x'}^{k_A}.$$

From (1.3), the support of $\mu_{x'}^{k_A}$ *is contained in the convex hull of* $W_A.x'$ *(the* W_A*-orbit of* x'*). Furthermore, by Theorem A, it is invariant under the action of the group* W_A *and contains the whole orbite* $W_A.x'$.

• *Let* $x\in\mathbb{R}^d$ *and* $r>0$. *According to (1.5) and (3.5) the harmonic kernel is given by*

$$h_k(r,x,y)=\int_{\mathbb{R}^d}\mathbf{1}_{[0,r]}\big(\sqrt{\|x''-y''\|^2+\|x'\|^2+\|y'\|^2-2\langle x',z'\rangle}\big)d\mu_{y'}^{k_A}(z'),\quad y\in\mathbb{R}^d.$$

The support of $h_k(r,x,.)$ *takes the following form*

$$supp\ h_k(r,x,.)=x''+B^{W_A}(x',r)=x''+\cup_{g\in W_A}B(gx',r)=\cup_{g\in W_A}B(gx,r).$$

Example 3.1 *Let* (e_1,e_2) *be the canonical basis of* $\mathbb{R}^2$. *Then, the set* $R:=\{\pm e_1,\pm e_2\}$ *is a root system in* $\mathbb{R}^2$, *its Coxeter-Weyl group is* $\mathbb{Z}_2^2$ *and the multiplicity function can be identified to a pair* $k=(k_1,k_2)$, *with* $k_i=k(e_i)\geq 0$, $i=1,2$. *Take* $x=(x_1,x_2)\in\mathbb{R}^2$ *with* $x_1,x_2>0$. *In this case, according to [16], Rösler's measure is given by* $\mu_x^k=\mu_{x_1}^{k_1}\otimes\mu_{x_2}^{k_2}$, *where* $\mu_{x_i}^{k_i}=\delta_{x_i}$ *if* $k_i=0$ *and*

$$\langle\mu_{x_i}^{k_i},f\rangle=\frac{\Gamma(k_i+1/2)}{\sqrt{\pi}\Gamma(k_i)}\int_{-1}^1 f(tx_i)(1-t)^{k_i-1}(1+t)^{k_i}dt.$$

if $k_i>0$ *(see [2]).*

• *If* $k=(0,0)$, $\mu_x^k=\delta_x$ *and* $h_k(r,x,y)=\mathbf{1}_{B(x,r)}(y)$.

• *If* $k=(k_1,0)$ *with* $k_1>0$, *then supp* μ_x^k *is the line segment between* x *and* $\sigma_{e_1}x=(-x_1,x_2)$ *and*

$$supp\ h_k(r,x,.)=B(x,r)\cup B(\sigma_{e_1}x,r).$$

• *If* $k_1,k_2>0$, *the support of* μ_x^k *is the convex hull of* $\mathbb{Z}_2^2.x$ *and the closed Dunkl ball is given by*

$$B^{\mathbb{Z}_2^2}(x,r)=supp\ h_k(r,x,.)=B\big((x_1,x_2),r\big)\cup B\big((-x_1,x_2),r\big)\cup B\big((x_1,-x_2),r\big)\cup B\big((-x_1,-x_2),r\big).$$

References

[1] C. F. Dunkl. *Differential-difference operators associated to reflection groups.* Trans. Amer. Math. Soc., 311, (1989), 167-183.

[2] C. F. Dunkl. *Integral kernels with reflection group invariance.* Canad. J. Math., 43, (1991), 123-183.

[3] C. F. Dunkl. *Hankel transforms associated to finite reflection groups.* Contemp. Math., 138, (1992), 123-138.

[4] C. F. Dunkl and Y. Xu. *Orthogonal Polynomials of Several variables.* Cambridge Univ. Press (2001).

[5] P. Etingof. *A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups.* Math. Res. Lett. 17 (2010), no. 2, 275-282.

[6] L. Gallardo and C. Rejeb. *A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications.* Trans. Amer. Math. Soc., Vol. 368, Number 5, May 2016, p.3727-3753 (electronically published on May 22, 2015, DOI:http://dx.doi.org/10.1090/tran/6671).

[7] J. E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, (1990).

[8] M. F. de Jeu. *The Dunkl transform.* Invent. Math., 113, (1993), 147-162.

[9] R. Kane. *Reflection Groups and Invariant Theory.* CMS Books in Mathematics. Springer-Verlag, New York, (2001).

[10] E. M. Opdam. *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group.* Compositio Math. 85 (1993), no. 3, 333-373.

[11] M. Rösler. *Positivity of Dunkl's intertwining operator.* Duke Math. J., 98, (1999), 445-463.

[12] M. Rösler. *Short-time estimates for heat kernels associated with root systems.* Special functions (Hong Kong, 1999), 309-323, World Sci. Publ., River Edge, NJ, 2000.

[13] M. Rösler and M. F. de Jeu. *Asymptotic analysis for the Dunkl kernel.* J. Approx. Theory, 119, (2002), no. 1, 110-126.

[14] M. Rösler. *Dunkl Operators: Theory and Applications.* Lecture Notes in Math., vol.1817, Springer Verlag (2003), 93-136.

[15] K. Trimèche. *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual.* Integ. Transf. and Spec. Funct., 12(4), (2001), 394-374.

[16] Y. Xu. *Orthogonal polynomials for a family of product weight functions on the spheres.* Canad. J. Math., 49(1), (1997), 175-192.